Electrical Wave Filters Employing Crystals with Normal and Divided Electrodes

By W. P. MASON and R. A. SYKES

I. Introduction

IN SEVERAL previous papers [1,2,3,4] the application of piezoelectric crystals to electric wave filters has been discussed. The underlying principles and some of the design procedures were given. These filters have received wide application in carrier telephone systems and radio systems both in the United States and abroad.[5] It is the purpose of the present paper to discuss more completely all the standard types of filters with crystals, and methods for determining their constants and attenuation characteristics. In addition some of the newer results for simplifying such filters are given.

The use of a divided plate crystal for filters resulted in cutting the number of crystals in half as was pointed out in three former papers.[2,3,4] The theory of the use of such crystals is discussed in this paper and an equivalent circuit is given for a crystal with two sets of plates. The application of this circuit to unbalanced filters allows the results for balanced lattice filters to be realized for unbalanced filters. For one connection of the two plates the resonance of the crystal can be made to appear in one arm of the equivalent lattice, while for the reverse connection the resonance appears in the other arm of the lattice.

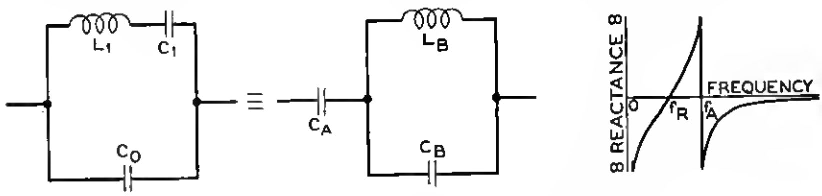
II. Crystal Filter Sections Which Can Be Realized in Lattice Networks

As pointed out in a previous paper [1] the most general filter characteristics for networks employing crystals can be realized in a lattice network, since every known form of a network can be reduced to a

[1] "Electrical Wave Filters Employing Quartz Crystals as Elements," W. P. Mason, *B. S. T. J.*, July 1934, pp. 405–452.
[2] "Resistance Compensated Band Pass Crystal Filters for Unbalanced Circuits," W. P. Mason, *B. S. T. J.*, Oct. 1937, pp. 423–436.
[3] "The Evolution of the Crystal Wave Filter," O. E. Buckley, *Jour. of Applied Physics*, Oct. 1936.
[4] "Crystal Channel Filters for the Cable Carrier Systems," C. E. Lane, *B. S. T. J.*, Vol. XVII, Jan. 1938, p. 125.
[5] "Channel Filters Employing Crystal Resonators," H. Stanesby and E. R. Broad, *P. O. E. E. Jour.*, 31, pp. 254–264, Jan. 1939.
[1] Loc. cit.

lattice network with realizable constants, whereas the converse is not necessarily true.

Let us consider first what types of filter characteristics can be obtained by using a crystal in one arm of a lattice network, and electrical or crystal elements in the other arm. As is well known the equivalent electrical network of a crystal is as shown in Fig. 1. The



$$C_A = C_0 + C_1\ ;\quad C_B = \frac{C_0}{C_1}(C_0 + C_1)\ ;\quad L_B = \frac{L_1}{\left(\frac{C_0 + C_1}{C_1}\right)^2}$$

$$f_R = \frac{1}{2\pi\sqrt{L_1 C_1}}\ ;\quad f_A = \frac{1}{2\pi}\sqrt{\frac{C_0 + C_1}{L_1 C_1 C_0}}$$

Fig. 1—Equivalent electrical circuit and reactance frequency characteristic of piezo-electric crystal.

element values, as calculated in a recent paper, for a plated crystal vibrating longitudinally are [6]

$$C_0 = \frac{K l_w l_y}{4\pi l_t} \times \frac{1}{9 \times 10^{11}} \text{ farads;} \qquad C_2 = \frac{S}{\pi^2}\frac{d'^{2}_{12}}{S'_{22}}\frac{l_w l_y}{l_t} \times \frac{1}{9 \times 10^{11}} \text{ farads;}$$

$$L_1 = \frac{\rho^{S_{22}^{2}} l_t l_y}{8 d'^{2}_{12} l_w} \times 9 \times 10^{11} \text{ henries,} \tag{1}$$

where l_y, l_w, l_t are respectively the length, width, and thickness of the crystal expressed in centimeters, K = specific inductive capacity, S_{22}' = inverse of Young's modulus along the direction of vibration, d_{12}' is the value of the piezo-electric constant along the direction of vibration, and ρ is the density of the crystal. The resistance depends on the clamping resistance, acoustic radiation from the ends of the crystal, internal damping losses, etc. In general the ratio of the reactance of the inductance L_1 to the resistance R at the resonant frequency f_R is from 20,000 to 300,000, depending on how the crystal is mounted, whether it is evacuated, etc. In general this resistance is so small that it can be neglected for design purposes, and only the ideal reactance characteristic need be considered.

[6] "A Dynamic Measurement of the Elastic, Electric and Piezoelectric Constants of Rochelle Salt," W. P. Mason, *Phys. Rev.*, Vol. 55, April 15, 1939, p. 775.

The reactance characteristic of the crystal, as shown in Fig. 1, is a negative reactance at low frequencies up to a resonant frequency f_R. For frequencies greater than f_R, the reactance becomer positive up to the anti-resonant frequency f_A, above which the reactance is again negative. The ratio of the anti-resonant frequency to the resonant frequency is determined directly by the ratio r of C_0 to C_1 existing in the crystal. As shown by Fig. 1,

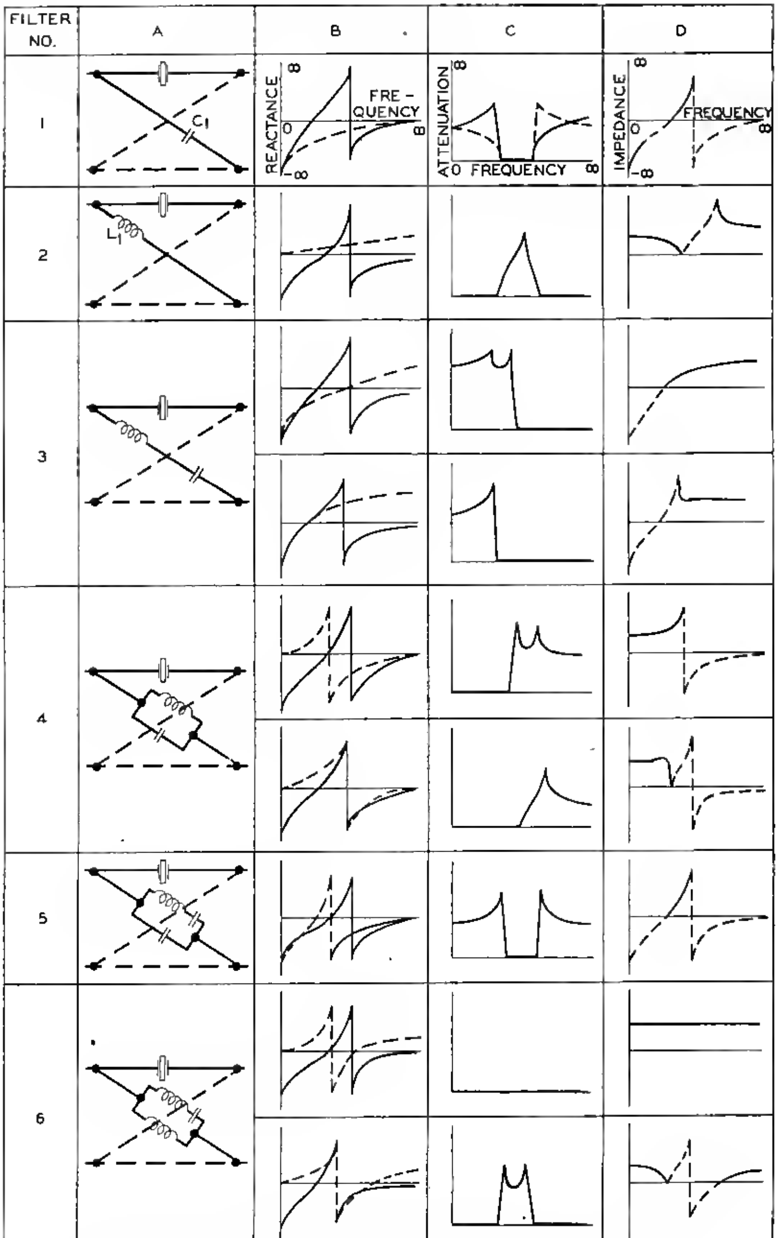
$$\frac{f_A}{f_R} = \sqrt{1 + \frac{1}{r}}. \tag{2}$$

This ratio is usually greater than 125 for a quartz crystal and hence the anti-resonant frequency is less than .4 per cent higher than the resonant frequency.

The previous papers considered mainly band-pass filters and discussed briefly low and high-pass crystal filters. It is also possible to obtain band elimination and all-pass crystal filters by combining electrical elements with the crystals in the proper manner. We consider, first, all the types of filters which can be obtained by using a single crystal in one arm of a lattice filter and electrical elements in the other arms. Figure 2 shows all the possible single-band characteristics which can be obtained by using a crystal in one arm and an electrical impedance, or crystal impedance, in the other lattice arm. For example, the first filter of the table shows a filter with a crystal in one arm and a capacitance in the other arm. Column B shows the reactance characteristic of each arm. A lattice filter will have a pass band when the reactances are of opposite sign and will attenuate when the reactances are the same sign. When the two reactances are equal the filter will have an infinite attenuation. This result follows from the expressions for the propagation constant and characteristic impedance of a balanced lattice network which are

$$\tanh\frac{P}{2} = \sqrt{\frac{Z_1}{Z_2}}; \qquad Z_0 = \sqrt{Z_1 Z_2}, \tag{3}$$

where Z_1 is the impedance of the series arm of the lattice and Z_2 that of the shunt arm. The third column shows the attenuation characteristic of this filter. It is a narrow band filter having a pass band between the resonant and anti-resonant frequencies of the filter. There is a peak of attenuation either above or below the band depending on the value of the capacitance C_1 in the lattice arm. The last column shows the value of the characteristic impedance of the filter as a function of the frequency. The dotted line indicates a



NOTE—IN COLUMN D, DOTTED LINES INDICATE REACTIVE IMPEDANCE
SOLID LINES INDICATE RESISTIVE IMPEDANCE

Fig. 2—Single band lattice filters employing a crystal in one arm.

reactance while a solid line indicates a resistance. In the pass band the filter has a resistive characteristic indicating a transmission of energy, while in the attenuating band the characteristic impedance is reactive indicating a reflection of energy.

Filter No. 2 shows what characteristic will be obtained if an ideal inductance is used in the lattice arm. As can be seen a band elimination filter will result with one attenuation peak. The width of the suppression band will be the separation between the resonant and anti-resonant frequencies.

The use of a series resonant circuit results in a high-pass filter as can be seen from filter No. 3. It is possible to obtain two dispositions of the resonant frequencies which will give a single pass band as shown by the two sets of curves. The first set gives a high-pass filter with two attenuation peaks and a simple characteristic impedance. The other arrangement gives one attenuation peak and a more complicated type of characteristic impedance. The theory of this balancing of characteristics obtainable with a lattice filter is well known,[7] and is useful, when it is necessary on account of reflection effects, to make the characteristic impedance constant nearly to the cutoff.

The use of an anti-resonant circuit results in a low-pass filter as shown by filter No. 4. Two characteristics are possible. Filters No. 5 and 6 show the characteristics obtainable by using series resonant circuits shunted by a capacitance or an inductance. In one case a band-pass filter with two peaks results, and in the other either a band suppression filter with two attenuation peaks or an all pass filter. It will be noted that the configurations used in the lattice arm of filter 5 is the equivalent circuit of the crystal and hence a crystal can be used in this arm. In fact the circuit is similar to one discussed in the former paper.[1]

Since the crystal positive reactance region is very narrow ($< .4\%$), all of the band pass and band elimination filters obtained by using a crystal in one arm will of necessity have very narrow band pass or band suppression regions. For high and low-pass filters the attenuation peaks will of necessity come close to the cutoff frequencies. In the all-pass structure the phase shift will be very sharp in the neighborhood of the crystal resonance. It was shown in the first paper,[1] that wider pass bands and more general characteristics can be obtained by employing inductance coils in series or parallel with the crystal. Figures 3 and 4 show the possible types of filters obtained by

[7] Cauer, Siebschattungen VDI, Verlag Berlin, 1931. H. W. Bode, "A General Theory of Electric Wave Filters," *Jour. of Math. and Physics*, Vol. XIII, pp. 275–362, Nov. 1934.

[1] Loc. cit.

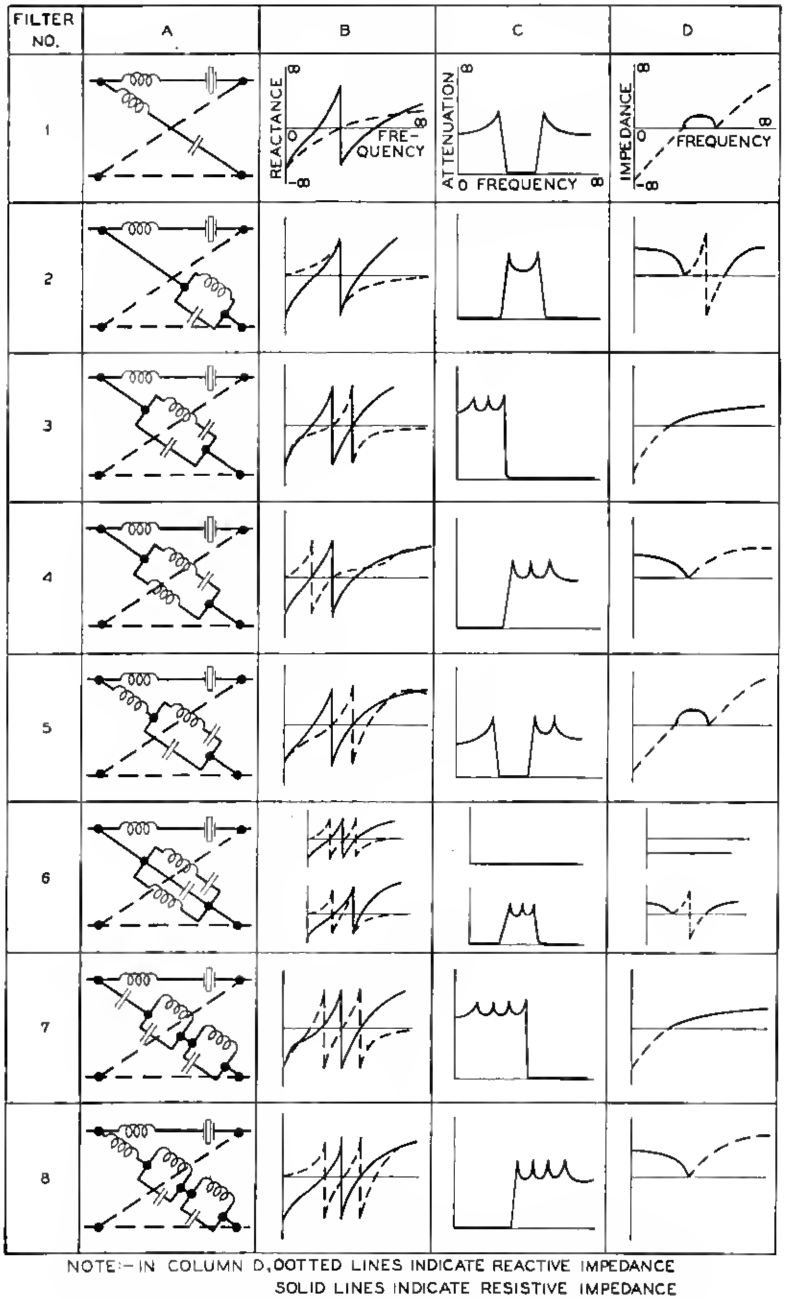
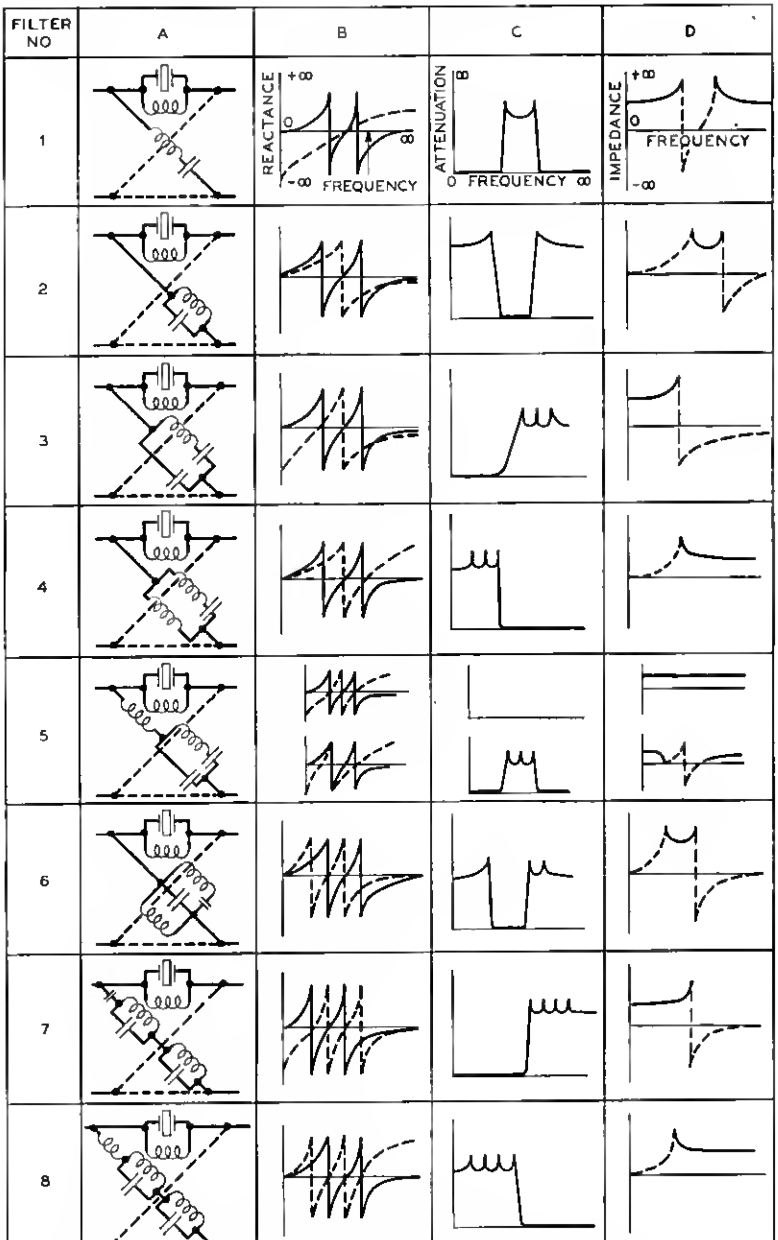


Fig. 3—Single band lattice filters employing a crystal and coil in series in one arm.



NOTE:- IN COLUMN D, DOTTED LINES INDICATE REACTIVE IMPEDANCE
SOLID LINES INDICATE RESISTIVE IMPEDANCE

Fig. 4—Single band lattice filters employing a crystal and coil in parallel in one arm.

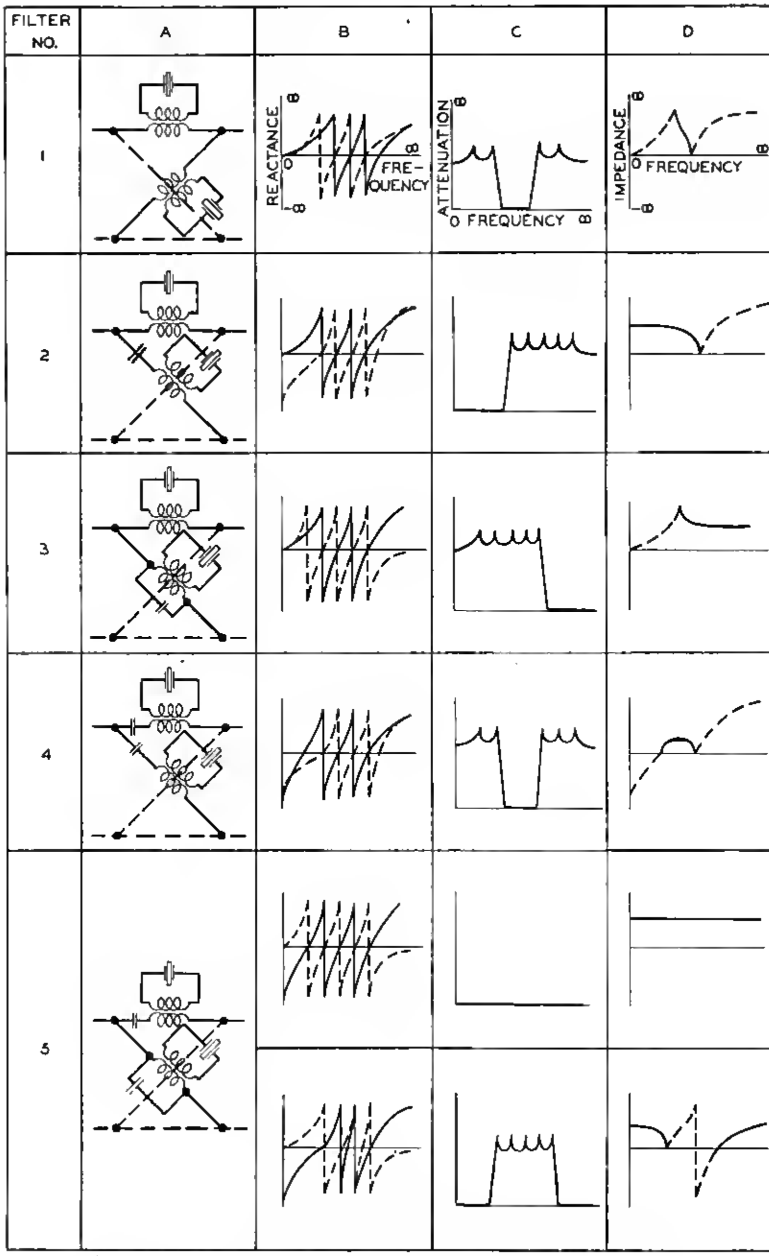
using a crystal and coil in one arm of the lattice and electrical elements in the other. Band-pass, band elimination, high and low-pass, and all-pass filters result. Only the simplest combinations of resonant frequencies giving the highest amount of attenuation are shown. As in filters 4 and 6 of Fig. 2, some of the anti-resonances and resonances of the two arms may be made to coincide giving filters with less attenuation but more flexibility in the impedance characteristics. Condensers can be incorporated in parallel or series with the crystals without affecting the type of characteristic obtained. This procedure is useful in controlling the widths of the pass or attenuation bands and in controlling the position of the peak values. In a number of the filters of Figs. 3 and 4, the equivalent circuit of the crystal occurs in the electrical circuit in the lattice arms. In these filters, crystals can also be used in the lattice arms. Filters 1 and 5 of Fig. 3 and filters 2 and 6 of Fig. 4 are band-pass filters which have been discussed in detail in former papers.[1, 2]

Crystals may also be used in more complicated electrical circuits, for example with transformers as shown in Fig. 5. This figure shows high, low, band-pass, band elimination and all-pass filters which can be constructed by employing transformers and crystals in each arm. More complicated structures still using single crystals can also be constructed but they tend to be of less importance since the dissipation introduced by the electrical elements neutralizes any benefit of using crystals.

It is possible, however, to use more crystals than one in one arm of a lattice and obtain filters having higher insertion losses outside the band without introducing more loss due to the electrical elements in the band. Figure 6 shows a number of such combinations with and without coils. The result of adding an additional crystal in one arm of a lattice is to add another elementary section of the type discussed in Appendix I. An example [8] of the characteristic obtainable with a band-pass filter with two crystals in each arm is shown on Fig. 7.

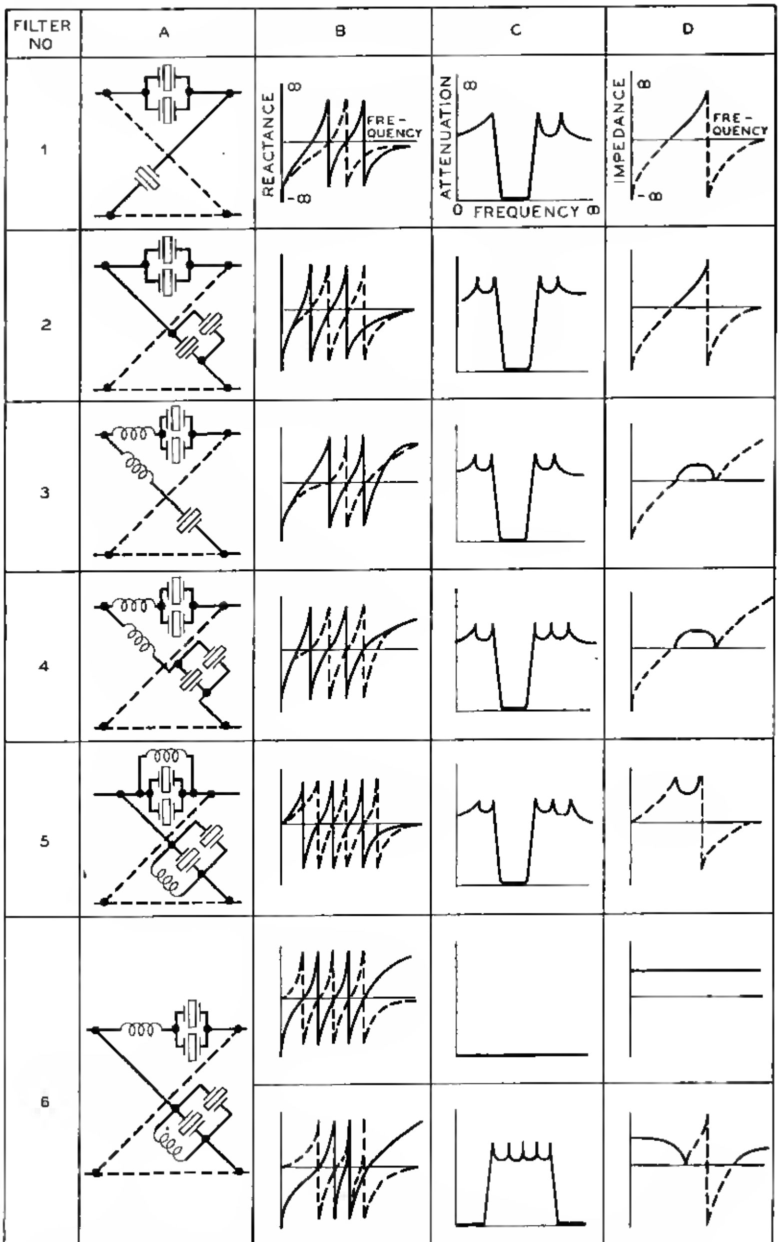
All of the filters discussed above were assumed to be constructed from dissipationless elements. When coils are used, however, a certain amount of resistance is associated with them which may alter the characteristics obtainable. As has been pointed out previously,[1] if the dissipation associated with the coils can be brought out to the ends of the arm either in series or parallel with the complete arm the effect of these resistances will be to add a constant loss independent of

[1, 2] Loc. cit.
[8] This filter was constructed and tested by Mr. H. J. McSkimin.
[1] Loc. cit.



NOTE - IN COLUMN D, DOTTED LINES INDICATE REACTIVE IMPEDANCE
SOLID LINES INDICATE RESISTIVE IMPEDANCE

Fig. 5—Single band lattice filters employing transformers and crystals.



NOTE - IN COLUMN D, DOTTED LINES INDICATE REACTIVE IMPEDANCE
SOLID LINES INDICATE RESISTIVE IMPEDANCE

Fig. 6—Single band lattice filters employing more than one crystal in the impedance arms.

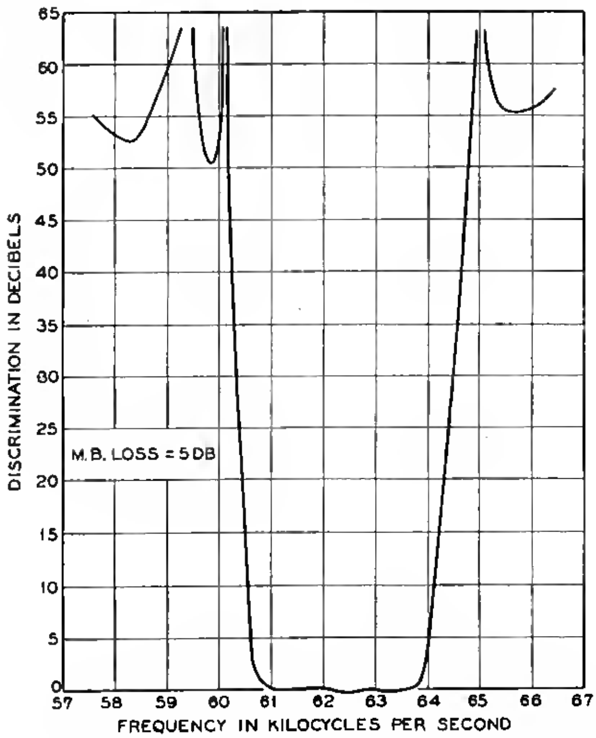


Fig. 7—Insertion loss characteristic of a single section band pass filter employing two crystals in each arm.

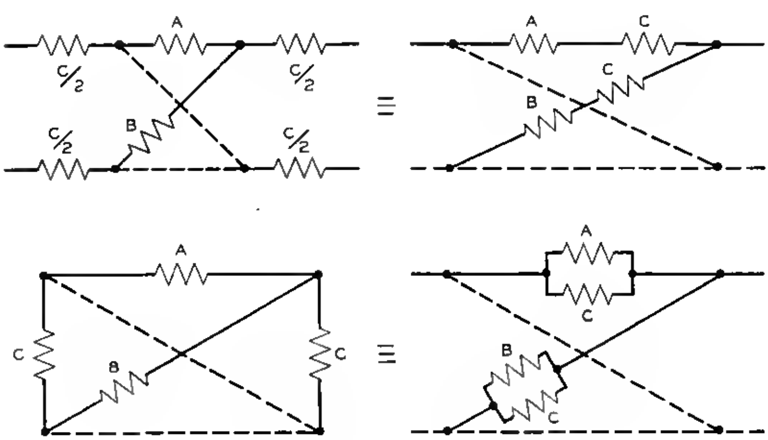


Fig. 8—Equivalences between lattice networks.

the frequency, and hence the discrimination obtainable will not be affected. This follows from the network equivalences shown in Fig. 8 which were first proved in a previous paper.[1] Even if this resistance compensation cannot be completely obtained it can often be obtained over a limited range near the cutoff and the peak frequencies by adding resistances to some of the crystal or electrical elements of such a value that the resistive components of the two arms are nearly equal over a limited frequency range. This results in cutting down the distortion near the cutoff and increasing the loss in the attenuated regions.

The lattice filters of Figs. 2 to 6 can be realized in ladder or bridge T forms in certain cases. If the two arms have two common series elements, then by the first equivalence of Fig. 8 they can be taken outside the lattice. Similarly, if two common shunt elements can be found in the two arms, then, by the second theorem of Fig. 8, the

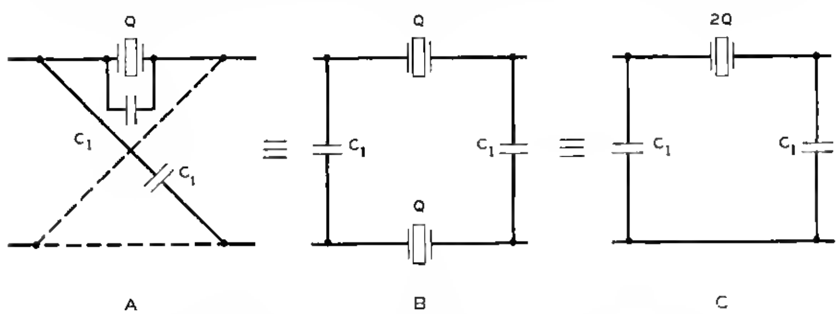


Fig. 9—Method for reducing a lattice filter to a π network filter.

elements can be placed in shunt on the ends of the filter. For example, suppose that we consider filter No. 1 of Fig. 2 and shunt the crystal by a capacitance C_1 which is equal to the capacitance of the lattice arm as shown in Fig. 9A. Then the two capacitances can be taken out in shunt leaving a crystal in the series arm of a π network as shown in Fig. 9C. This has the same type of characteristic as the lattice but considerably greater limitations.

A somewhat more general transformation can be made to a bridge T network of the type shown in Fig. 10A. This network is equivalent to the lattice network shown in Fig. 10B. As is evident, if we have an impedance in parallel with one arm and in series with the other, the resulting lattice can be transformed into a bridge T network. For example, in Fig. 3, filter No. 2, if we reverse the lattice and series arms, which can be done without changing the characteristics except for a 180° phase reversal, the filter can easily be reduced to a bridge

T network as shown in Fig. 11. The shunt coil L_0 is usually considerably smaller than the series coil L_1 so that L_1 can be divided into two coils, L_0 and $L_1 - L_0$. The transformation then becomes as

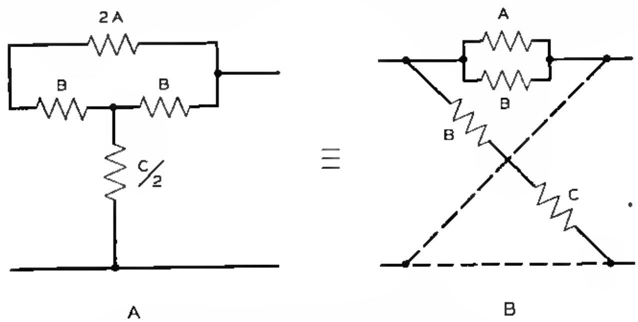


Fig. 10—Equivalence between bridge T and lattice networks.

shown in Fig. 11 with the element values shown. $Q/2$ indicates that the impedance of the crystal in the shunt arm is half that in the lattice arm. This transformation is applicable particularly to low and high-pass filters and band elimination filters.

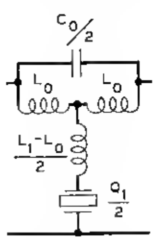


Fig. 11—A bridge T band elimination crystal filter.

Another transformation which can be employed is that for a three-winding transformer, for, as shown in a previous paper,[2] a three-winding transformer connected to two impedance arms as shown in Fig. 12 is equivalent to a transformer and a lattice filter with small

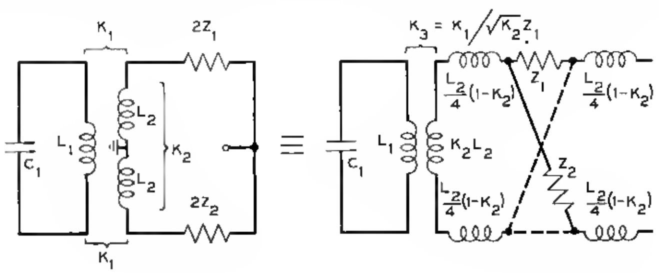


Fig. 12—Lattice equivalent of a three-winding transformer.

[2] Loc. cit.

coils on the ends. By making the coupling in the secondary high these coils can be made very small and can usually be neglected.

Another method for reducing balanced lattice filters to unbalanced circuits is to employ crystals with two sets of plates as described in section IV.

III. Method for Calculating the Element Values of the Filter

The curves in Figs. 2 to 6 give a qualitative picture of what type of characteristics can be obtained by the use of crystals in filter networks. In order to determine what band widths and dispositions of attenuation peaks are realizable with crystals it is necessary to calculate the element values, since a crystal cannot be made with a ratio of capacitances under 125.

The actual process of calculation can be divided into two parts. The first part consists in a determination of the critical frequencies of the arms of the network in terms of the desired attenuation characteristic. The second part consists in calculating the element values from the critical frequencies by means of Foster's theorem.

The attenuation characteristics obtainable with filters are discussed in Appendix I, and it is there shown that the attenuation characteristic of a complicated filter structure can be regarded as the sum of the attenuation characteristics of a number of elementary filters. The critical resonant frequencies of the filter are evaluated in terms of the cutoff frequencies and the position of the attenuation peaks with respect to the cutoff frequencies. The ratios of the impedances of the two arms at zero or infinite frequencies are evaluated in terms of the network parameters. With the aid of these equations, and Foster's theorem discussed below, the element values can be evaluated for any desired attenuation characteristic. Whether the characteristic is realizable or not depends on whether the element values of the equivalent circuit of the crystal calculated have a low enough ratio of capacitances to be realized in practice. The actual value of the series capacitance C_1 of the equivalent circuit of the crystal shown in Fig. 1 may also be too large to be physically realizable.

Having obtained the critical frequencies by the calculations given in the appendix, the element values can be calculated by using Foster's theorem. Foster's theorem [9] deals with impedances in the form of a number of series resonant circuits in parallel as shown on Fig. 13A or a number of antiresonant circuits in series as shown on Fig. 13B.

[9] See "A Reactance Theorem," *B. S. T. J.*, April 1924, page 259.

In either case the impedance of the network can be written in the form

$$Z = -jH\left[\frac{\left(1-\frac{\omega^2}{\omega_1^2}\right)\left(1-\frac{\omega^2}{\omega_3^2}\right)x\cdots x\left(1-\frac{\omega^2}{\omega_{2n-1}^2}\right)}{\omega\left(1-\frac{\omega^2}{\omega_2^2}\right)x\cdots x\left(1-\frac{\omega^2}{\omega_{2n-2}^2}\right)}\right], \tag{4}$$

where $H \geqq 0$ and $0 = \omega_0 \leqq \omega_1 \leqq \cdots \leqq \omega_{2n-1} \leqq \omega_{2n} = \infty$. For the series resonant circuits of Fig. 13A, the element values are given by

$$L_i = \frac{1}{C_i\omega_i^2} = \left(\lim_{\omega\to\omega_i}\right)\left(\frac{j\omega Z}{\omega_i^2-\omega^2}\right); \quad i = 1, 3, \cdots 2n-1. \tag{5}$$

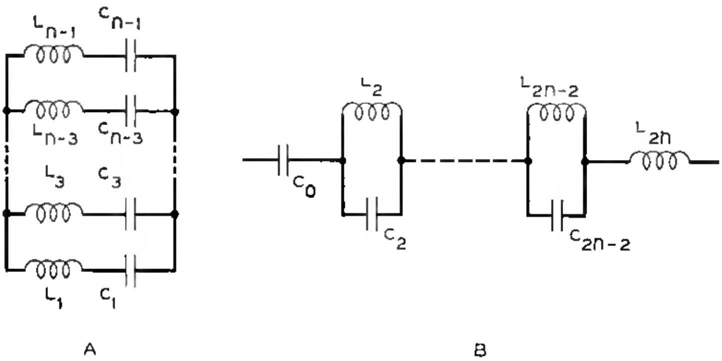


Fig. 13—Impedances arranged in form for application of Foster's Theorem.

For the antiresonant circuits in series the values become

$$C_i = \frac{1}{L_i\omega_i^2} = \left(\lim_{\omega\to\omega_i}\right)\left(\frac{j\omega}{Z(\omega_i^2-\omega^2)}\right); \quad i = 0, 2, 4, \cdots 2n. \tag{6}$$

These values include the limiting values for the series case of Fig. 13B.

$$C_0 = \frac{1}{H}; \quad L_0 = \infty, \quad C_{2n} = 0; \quad L_{2n} = \frac{H(\omega_2^2 w_4^2 \cdots \omega_{2n-2}^2)}{(\omega_1^2\omega_3^2 \cdots \omega_{2n-1}^2)}. \tag{7}$$

Hence if the elements of one arm of the lattice are arranged in either of the forms shown on Figs. 13A or B, the element values can be calculated from equations (5) and (6).

If they are not in this form, they can be transformed into one of these two forms by well known network transformations. For example, all the filters of Figs. 2, 3 and 4 are of this form or can be put in this form by employing the simple network transformation of Fig. 1. For the two crystal sections shown on Fig. 6, the series

inductance can be evaluated by equation (7) and subtracted from the impedance Z. This leaves an impedance

$$Z' = Z - j\omega L, \tag{8}$$

from which can be evaluated the constants of the two crystals in parallel by employing equation (6). In this way the constants of any filter can be evaluated when the desired attenuation and impedance characteristic are specified. Several of the band-pass filters are discussed in detail in a former paper.[2]

IV. Application of Divided Plate Crystals to Balanced and Unbalanced Filters

The use of a divided plate crystal to cut the number of crystals in half in a balanced lattice filter has been mentioned previously.[2, 3, 4, 10] The theory of this use has not been previously discussed and since it results in further applications it seems worth while to present it here.

In order to use the divided plate crystal in filters it is necessary to find an equivalent circuit for such a crystal which will hold for measurements between any pair of the four terminals. It was shown in a previous paper [6] that an electro-mechanical equivalent of a fully plated crystal free to vibrate on both ends could be represented as shown in Fig. 14A. In this figure the capacitance C_0 is the static capacitance of the crystal, the condenser C_M represents the effective compliance of the crystal at the resonant frequency, and the inductance L_M represents the effective mass. A perfect transformer of impedance ratio 1 to φ^2, where

$$\varphi^2 = \frac{(d_{12}l_w)^2}{(s_{22'})} \tag{9}$$

represents the coupling from electrical to mechanical energy. φ in effect is the ratio of the force exerted by the crystal when it is clamped, to the applied voltage or it is the force factor of the system. If now we use only half the plating on the crystal, for example the plates 1, 2 of Fig. 14B, the same representation will hold. The static capacitance C_0 will be divided by 2, and the force applied by a given voltage will also be divided by 2 or the transformer ratio will be $\varphi/2$. The same compliance and mass will be operative. Hence the equivalent circuit of a crystal with plates covering half the crystal will be as shown on Fig. 14C. For a crystal with two sets of plates, the repre-

[2, 3, 4] Loc. cit.
[10] See patent 2,094,044, W. P. Mason, issued Sept. 28, 1937.
[6] Loc. cit.

sentation shown on Fig. 14D can be used if we are interested only in the transmission from one set of plates to the other. The numbering on the terminals agrees with that shown on Fig. 14B and is necessary in order that a given voltage E will produce the same displacement in the crystal when the voltage is applied between 1 and 2 and 3 and 4.

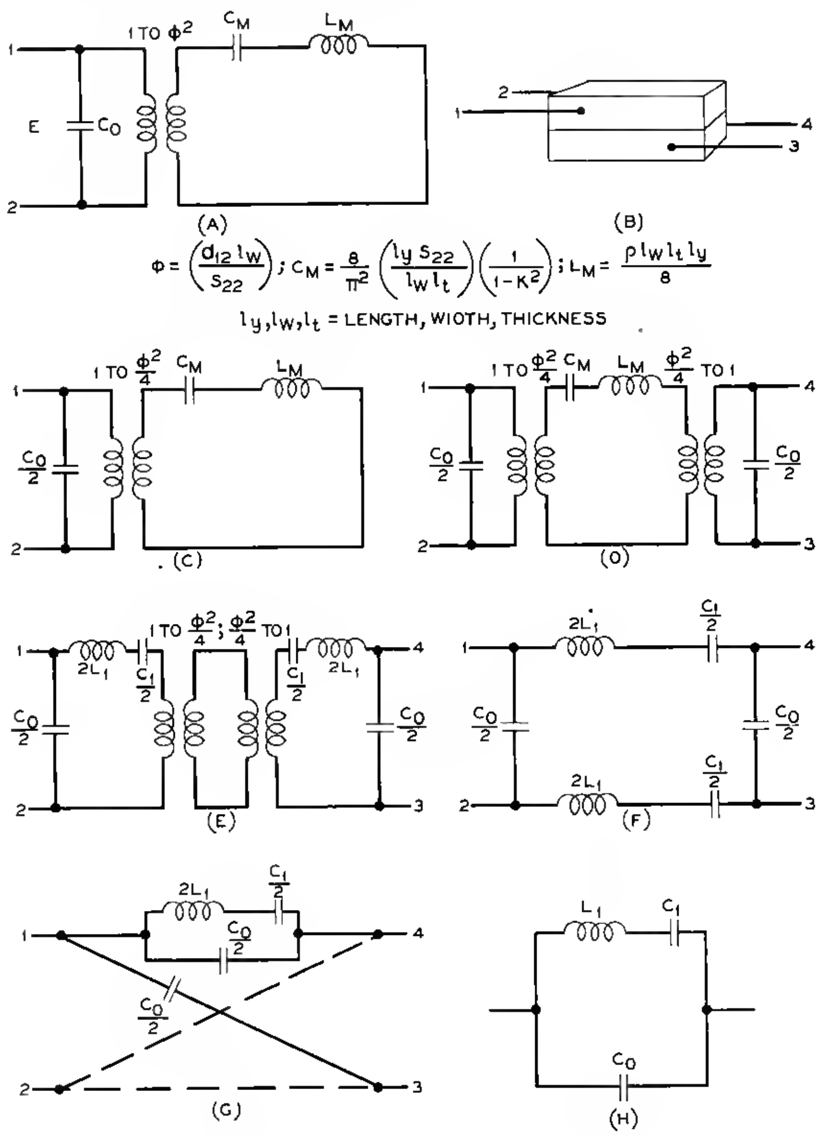


Fig. 14—Equivalent networks to represent transmission through divided plate crystal.

For purely electrical measurement, we can get rid of the two ideal transformers by taking half the impedance of C_M and L_M through each transformer as shown in Fig. 14E. Since we have left two opposing transformers of equal ratio they can be eliminated and the network of Fig. 14F results. This is shown in balanced form. Figure 14G shows the same network expressed in lattice form which is easily done by using the equivalences of Fig. 8. This represents a crystal of twice the impedance of the fully plated crystal in each series arm of the lattice with the static capacitances C_0 in the other arm. If we connect terminal 1 to 3 and 4 to 2, or in other words we use a completely plated crystal, the equivalent circuit reduces to that for a fully plated crystal as shown in Fig. 14H.

The networks of Fig. 14, F and G, represent the two plate crystals for transmission through the crystal, but do not give a four-terminal equivalence. For example, if we measure the crystal between terminals 1 and 3 we should not expect any impedance due to the vibration of the crystal since there is no field applied perpendicular to the thickness. The representation of Fig. 14, F or G, would not indicate this. The same sort of problem arises when it is desirable to obtain a four-terminal representation of a transformer and can be solved by using a lattice network representation with positive and negative inductance elements. The same procedure can be employed for a crystal and the steps are shown in Fig. 15.

We start with the lattice representation of Fig. 14G but employ the series form of the impedance of a crystal shown in Fig. 1. The series capacitance is divided into two parts, $C_0/2$ and a negative capacitance necessary to make the total series capacitance equal to C_0 plus C_1. This negative capacitance and the antiresonant circuit are lumped as one impedance $2Z$ in Fig. 15B. Now by the network equivalence of Fig. 8, we can take the series capacitances $C_0/2$ outside the network. We can also add an impedance $Z/2$ on the ends of the network provided we add a negative Z in series with all arms of the network as shown in Fig. 15C. The network of Fig. 15C is equivalent to that of Fig. 15A as far as transmission through it is concerned, but is different if we measure impedances between any of the four terminals. For example, if we measure the impedance between the terminals 1 and 4, the impedance of the network reduces to that shown in Fig. 15D. The impedance of the parallel circuit reduces to a plus Z shunted by a minus Z which introduces an infinite impedance. Similarly between 1 and 4, 2 and 3, and 2 and 4 the impedance becomes infinite as it should be if we neglect the small static capacitances existing in the crystal. If we take account of these the complete

four-terminal representation of a crystal becomes that shown in Fig. 15E. Ordinarily the capacitances C_{13}, C_{14}, C_{23}, C_{24} are small enough to be neglected. Figure 15E then is a complete equivalent circuit for a two-plate piezo-electric crystal which is valid for any kind of impedance or transmission measurements.

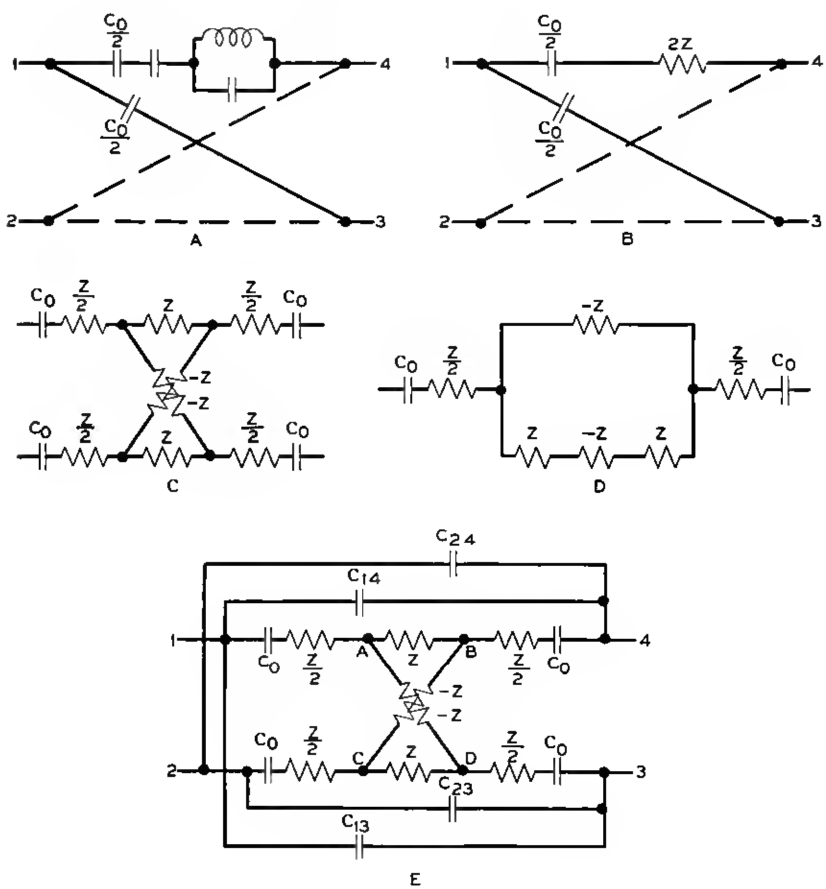


Fig. 15—Four-terminal equivalent network for divided plate crystal.

There are four possible connections for a crystal with two sets of plates used in a balanced filter. These connections and their equivalent circuits for transmission through as used in the filter are shown in Fig. 16. In order to prove these equivalences let us consider the equivalence shown on Fig. 16A. The four-terminal network representation for this case is shown in Fig. 17, which is obtained from

Fig. 15E. The capacitances C_{13} and C_{24} will be equal due to the symmetry in the crystal, while C_{14} will equal C_{23} for the same reason. These capacitances are directly connected to the outside terminals

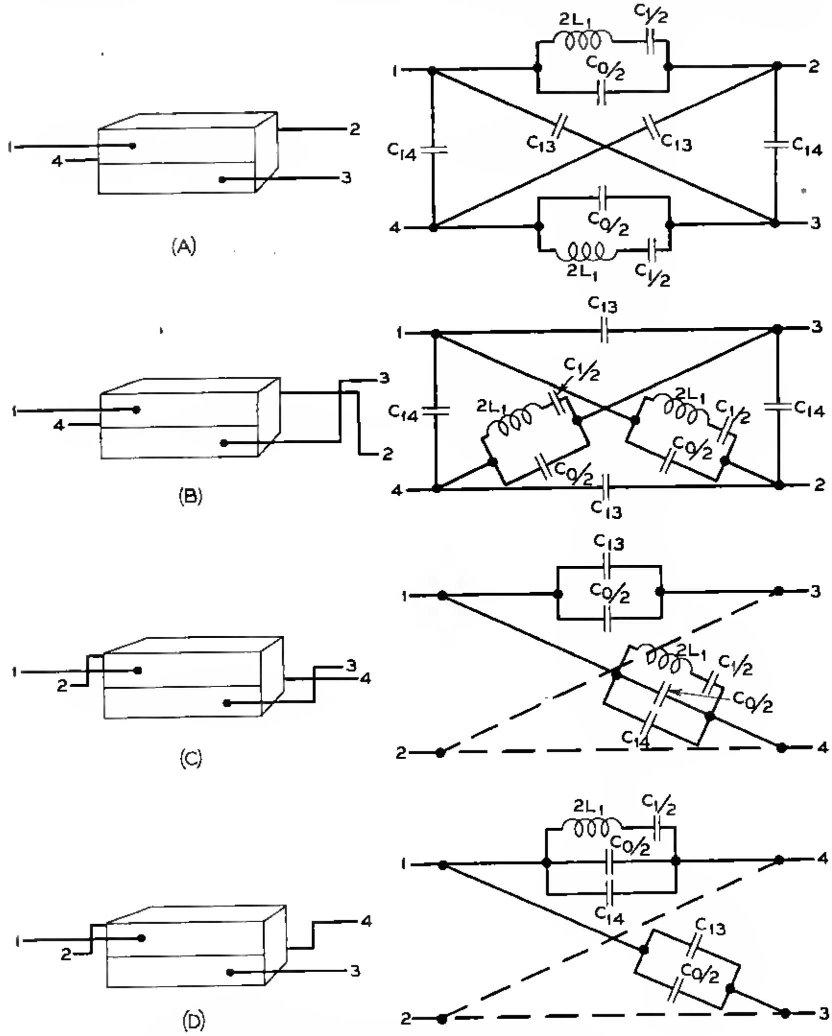


Fig. 16—Balanced divided plate crystal connections and their equivalent lattice electrical circuits.

and hence in obtaining the equivalent lattice they can be connected in directly. The remainder of the circuit can be reduced to its equivalent lattice by employing the equivalence shown in Fig. 8. Taking in the parallel impedance Z and the series impedances $Z/2 + (-j/\omega C_0)$, the

network of Fig. 17B results. On account of the paralleling of the $-Z$ and $+Z$ the lattice arm vanishes and the network reduces to that shown in Fig. 16A. In a similar manner the other equivalences result.

The use of divided plating crystals to obtain wide band filters by using series coils to widen the band is obvious. If we connect two

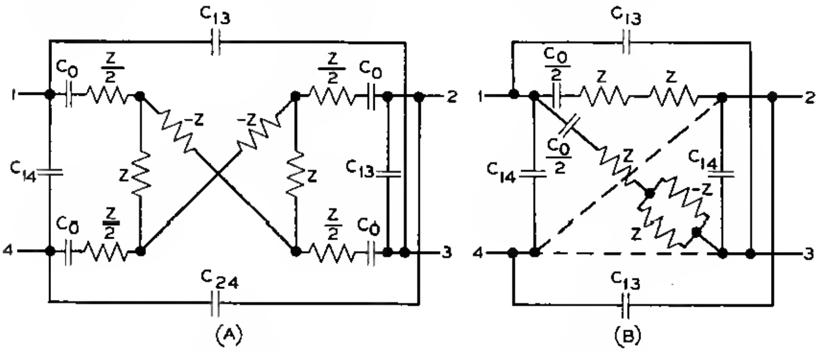


Fig. 17—Method for proving equivalence of Fig. 16A.

crystals as shown in Fig. 18A, one crystal being connected as shown in Fig. 16A and the other in Fig. 16B, a lattice filter equivalent to that is shown in Fig. 18B. In the series arms we have a crystal of twice the impedance of the fully plated crystal Q_1 shunted by the

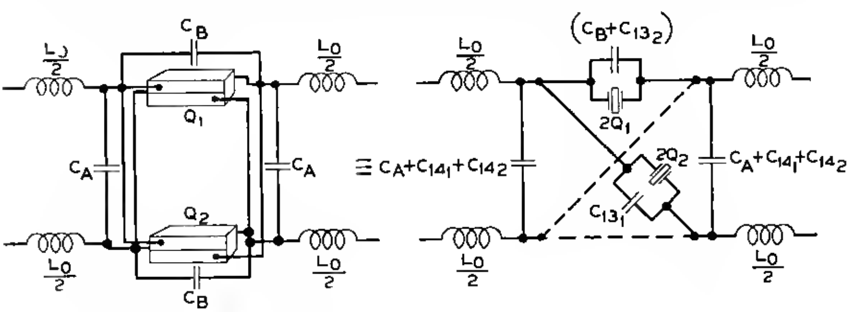


Fig. 18—Band pass crystal filter employing connections of Figs. 16A and B.

capacitance C_B and the capacitance C_{13} of the second crystal Q_2. In the lattice arms we have a crystal of twice the impedance of the fully plated crystal Q_2 in parallel with the capacitance C_{13} of the crystal Q_1. On the ends of the lattice we have capacitances $C_A + C_{14_1} + C_{14_2}$. It is obvious, then, that by using divided plate crystals we can replace two identical crystals in the two arms with crystals having twice the impedance of the fully plated crystals. This result can be

utilized in any type of filter where two crystals occur in series or lattice arms of a balanced lattice filter.

The connections C and D of Fig. 16 can also be used to give wide band filters but on account of the extra capacitance shunting each crystal, as wide bands cannot be obtained. This is shown in Fig. 19 which shows two crystals connected as shown in Figs. 16C and D used in a filter. Since each crystal is shunted by half the static capacitance of the other, the ratio of capacitances will be about twice that in the connection shown in Fig. 18 and the band width possible will be about 70 per cent of that shown in Fig. 18. Hence the connections of Fig. 18 are usually desirable.

The connections of 16C and D can be duplicated in unbalanced form as shown in Fig. 20. These equivalents are easily worked out from the network of Fig. 15E by employing Bartlett's theorem. An

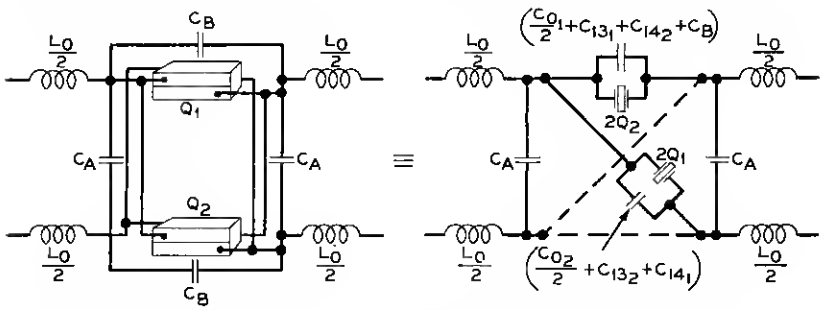


Fig. 19—Band pass crystal filter employing connections of Figs. 16C and D.

unbalanced filter of the type shown in Fig. 19 can be obtained by combining the two connections shown in Fig. 20 as shown in Fig. 21. It will be noted that across the series arm we have a capacitance $C_B + 2C_{14_1} + 2C_{13_2} + C_{0_2}/2$ while the lattice arm has only the capacitance $C_{0_1}/2$. To get attenuation peaks which are separated from the pass band by a large frequency range it is necessary to keep the capacitances C_{14_1} and C_{13_2} small. This can be accomplished by using shielding strips on the plating as shown in Fig. 22 for the two types of connection. In the B connection, the grounding strip is effectively obtained by making the grounded plates 2 and 3 slightly larger than 1 and 4. These grounding strips then act like a guard ring and reduce the stray capacitances.

The same process can be applied to any of the filters of Figs. 2 to 6 to obtain in unbalanced form the characteristics obtained in lattice form. In general the characteristics are somewhat more limited since

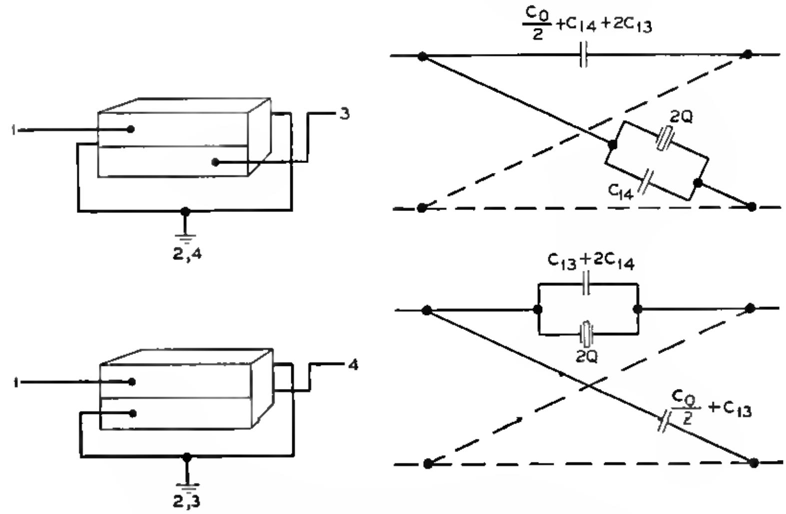


Fig. 20—Unbalanced divided plate crystal connections and their equivalent electrical circuits.

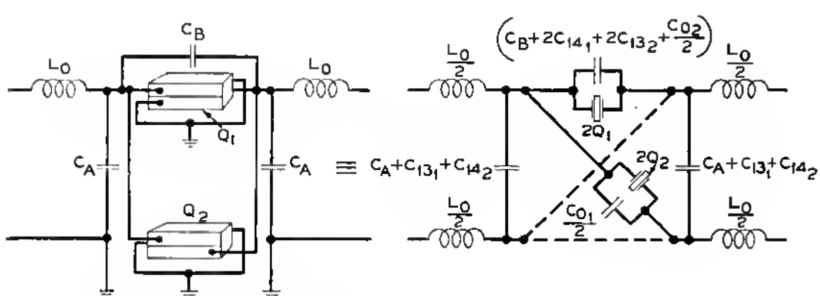


Fig. 21—Unbalanced band pass filter employing the connections of Fig. 20.

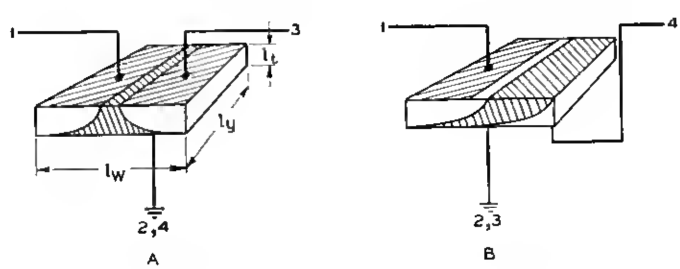


Fig. 22—Method for reducing stray capacitances in unbalanced filter connections.

in effect we have to use crystals with twice the ratio of capacitances than can be used in the balanced case.

APPENDIX

A Determination of the Resonant Frequencies of Lattice Filters

In order to obtain the element values of the filters shown in this paper it is necessary to determine the resonant frequencies of the elements in terms of the desired characteristics of the filter. It is the purpose of this appendix to show how these resonant frequencies may be derived.

The simplest type of band-pass filter section—referred to as the elementary section—is one in which there is one resonance in each arm of a lattice filter as shown in Fig. 23A. The impedance of the

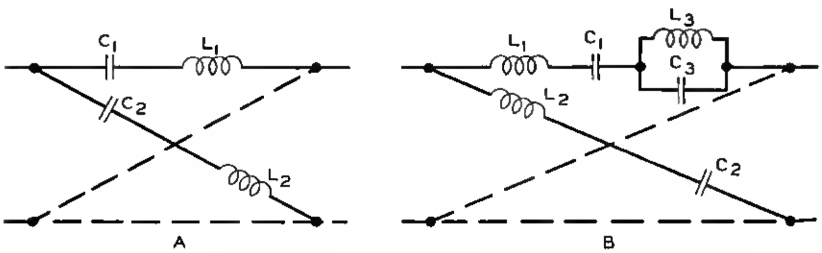


Fig. 23—Lattice filter configuration for elementary band pass sections.

series and lattice arms takes the form

$$Z_1 = \frac{-j}{\omega C_1}\left[1 - \frac{\omega^2}{\omega_A{}^2}\right]; \qquad Z_2 = \frac{-j}{\omega C_2}\left[1 - \frac{\omega^2}{\omega_B{}^2}\right], \tag{10}$$

where ω is 2π times the frequency f, f_A the resonant frequency of the series arm which also is the lower cutoff of the filter, and f_B the resonant frequency of the lattice arm which is also the upper cutoff.

The characteristic impedance and propagation constant are from equation (3)

$$Z_0 = \sqrt{Z_1 Z_2} = \sqrt{-\frac{1}{\omega^2 C_1 C_2}\left[1 - \frac{\omega^2}{\omega_A{}^2}\right]\left(1 - \frac{\omega^2}{\omega_B{}^2}\right)},$$

$$\tanh\frac{P}{2} = \sqrt{\frac{Z_1}{Z_2}} = \sqrt{\frac{C_2}{C_1}\left(\frac{1 - \omega^2/\omega_A{}^2}{1 - \omega^2/\omega_B{}^2}\right)} = m\sqrt{\frac{1 - \omega^2/\omega_A{}^2}{1 - \omega^2/\omega_B{}^2}}. \tag{11}$$

It is desirable to correlate the value of m with the frequency of infinite attenuation in the filter. Since the filter will have an infinite attenua-

tion when $\tanh P/2 = 1$, we have

$$m = \sqrt{\frac{1 - \omega_\infty^2/\omega_B^2}{1 - \omega_\infty^2/\omega_A^2}}, \tag{12}$$

where ω_∞ is $2\pi f_\infty$, where f_∞ is the frequency of infinite attenuation. For a single section since $m = \sqrt{C_2/C_1}$, m must be real and lie between 0 and infinity. The possible attenuation characteristics obtainable with the simple section can be calculated from equation (11). It will be noted that when $\omega = 0$

$$m = \tanh \frac{P_0}{2}. \tag{13}$$

Similar equations for low-pass, high-pass and all-pass filters can be derived from these equations by letting f_A go to zero or f_B to ∞, or both. These equations are:

For Low-Pass Filters

$$\tanh \frac{P}{2} = \lim \omega_A \to 0 \left(\sqrt{\frac{1 - \omega_\infty^2/\omega_B^2}{1 - \omega_\infty^2/\omega_A^2}}\sqrt{\frac{1 - \omega^2/\omega_A^2}{1 - \omega^2/\omega_B^2}}\right)$$

$$= \sqrt{1 - \omega_B^2/\omega_\infty^2}\sqrt{\frac{- \omega^2/\omega_B^2}{1 - \omega^2/\omega_B^2}} = m_l\sqrt{\frac{- \omega^2/\omega_B^2}{1 - \omega^2/\omega_B^2}}. \tag{14}$$

For High-Pass Filters

$$\tanh \frac{P}{2} = \lim \omega_B \to \infty \left(\sqrt{\frac{1 - \omega_\infty^2/\omega_B^2}{1 - \omega_\infty^2/\omega_A^2}}\sqrt{\frac{1 - \omega^2/\omega_A^2}{1 - \omega^2/\omega_B^2}}\right)$$

$$= \sqrt{\frac{1}{1 - \omega_\infty^2/\omega_A^2}}\sqrt{1 - \omega^2/\omega_A^2} = m_h\sqrt{1 - \omega^2/\omega_A^2}. \tag{15}$$

For All-Pass Filters

$$\tanh \frac{P}{2} = \begin{matrix}\lim \omega_A \to 0 \\ \lim \omega_B \to \infty\end{matrix} \left(\sqrt{\frac{1 - \omega_\infty^2/\omega_B^2}{1 - \omega_\infty^2/\omega_A^2}}\sqrt{\frac{1 - \omega^2/\omega_A^2}{1 - \omega^2/\omega_B^2}}\right)$$

$$= \sqrt{\frac{1}{- \omega_\infty^2}}\sqrt{- \omega^2}. \tag{16}$$

For this case, since there is no peak in the real frequency range, we must let ω_∞ be imaginary or $i\omega_\alpha$. Then

$$\tanh \frac{P}{2} = \frac{1}{\omega_\alpha}\sqrt{- \omega^2} = m_a\sqrt{- \omega^2}. \tag{17}$$

The band elimination filter cannot be obtained from the band-pass filter by a limiting process. For the simplest band elimination filter

with a single peak as shown on Fig. 2, filter 2, the equations are

$$Z_0 = \sqrt{\frac{L_1}{C_2}\left(\frac{1 - \omega^2/\omega_A^2}{1 - \omega^2/\omega_B^2}\right)};$$
$$\tanh\frac{P}{2} = \sqrt{\frac{-\omega^2 L_1 C_2(1 - \omega^2/\omega_B^2)}{(1 - \omega^2/\omega_A^2)}} = \frac{1}{\omega_\alpha}\sqrt{\frac{-\omega^2(1 - \omega^2/\omega_B^2)}{(1 - \omega^2/\omega_A^2)}}; \quad (18)$$
$$\omega_\alpha = \frac{1}{\sqrt{L_1 C_2}} = \sqrt{\frac{-\omega_\infty^2(1 - \omega_\infty^2/\omega_B^2)}{(1 - \omega_\infty^2/\omega_A^2)}}.$$

Hence when the position of the peak of infinite attenuation and the characteristic impedance Z_0 at zero frequency are specified L_1 and C_2 can be determined.

We next consider the case of a filter with a total of three resonances rather than two. For a band-pass filter this will be represented by the impedance arms shown on Fig. 23B. The impedance of the series and lattice arms will be

$$Z_1 = \frac{-j}{\omega C_1}\left[\frac{(1 - \omega^2/\omega_A^2)(1 - \omega^2/\omega_B^2)}{1 - \omega^2/\omega_2^2}\right]; \quad Z_2 = \frac{-j}{\omega C_2}\left(1 - \frac{\omega^2}{\omega_2^2}\right). \quad (19)$$

Combining these to form the propagation constant and characteristic impedance we find

$$Z_0 = \sqrt{\frac{-1}{\omega^2 C_1 C_2}(1 - \omega^2/\omega_A^2)(1 - \omega^2/\omega_B^2)},$$
$$\tanh\frac{P}{2} = \sqrt{\frac{C_2}{C_1}\frac{(1 - \omega^2/\omega_A^2)(1 - \omega^2/\omega_B^2)}{(1 - \omega^2/\omega_2^2)^2}} \quad (20)$$
$$= B\sqrt{\frac{(1 - \omega^2/\omega_A^2)(1 - \omega^2/\omega_B^2)}{(1 - \omega^2/\omega_2^2)^2}}.$$

We wish to show now that this type of section has an attenuation characteristic equal to that obtained by two sections of the kind shown in Fig. 23A. To show this we write

$$\tanh\frac{P_1 + P_2}{2} = \frac{\tanh\frac{P_1}{2} + \tanh\frac{P_2}{2}}{1 + \tanh\frac{P_1}{2}\tanh\frac{P_2}{2}}. \quad (21)$$

Substituting the value of $\tanh P/2$ given by equation (14) in (21) and letting the two cutoffs ω_A and ω_B coincide for the two sections, we have

$$\tanh\frac{P_1 + P_2}{2} = \frac{m_1 + m_2}{1 + m_1 m_2}\sqrt{\frac{(1 - \omega^2/\omega_A^2)(1 - \omega^2/\omega_B^2)}{(1 - \omega^2/\omega_2^2)^2}}, \quad (22)$$

where

$$\omega_2^2 = \frac{\omega_A^2\omega_B^2(1 + m_1 m_2)}{\omega_A^2 + \omega_B^2 m_1 m_2}. \quad (23)$$

Comparing (22) with (20) we see that

$$P = P_1 + P_2 \quad \text{and} \quad B = \frac{m_1 + m_2}{1 + m_1 m_2} = \tanh\left(\frac{P_{0_1} + P_{0_2}}{2}\right). \tag{24}$$

We see then that a section with three resonant frequencies can be made to have the same attenuation characteristic as the sum of two simple sections. It is, however, more general since in equations (23) and (24) real values of ω_2^2 and B can be obtained by taking

$$m_1 = m_{1_r} + i m_{1_i}; \qquad m_2 = m_{1_r} - i m_{1_i}; \tag{25}$$

that is, the parameter m_1 can be made complex if the second parameter m_2 is made its conjugate. Such complex sections can be made to have attenuation peaks which are finite even in the absence of dissipation.[7]

By letting $\omega_A \to 0$ or $\omega_B \to \infty$, the equivalent relations for low-pass, high-pass and all-pass filters can be obtained. These are

Low-Pass Filter

$$\tanh\frac{P}{2} = (m_1 + m_2)\sqrt{\frac{-(\omega^2/\omega_B^2)(1 - \omega^2/\omega_B^2)}{(1 - \omega^2/\omega_2^2)^2}};$$
$$\omega_2^2 = \frac{\omega_B^2}{1 + m_1 m_2}; \qquad m_1 = \sqrt{1 - \frac{\omega_{\infty 1}^2}{\omega_B^2}}. \tag{26}$$

High-Pass Filter

$$\tanh\frac{P}{2} = \frac{m_1 + m_2}{1 + m_1 m_2}\sqrt{\frac{(1 - \omega^2/\omega_A^2)}{(1 - \omega^2/\omega_2^2)^2}}; \quad \omega_2^2 = \frac{\omega_A^2\omega_B^2[1 + m_1 m_2]}{\omega_A^2 + \omega_B^2 m_1 m_2}. \tag{27}$$

All-Pass Filter

$$\tanh\frac{P}{2} = (m_1 + m_2)\sqrt{\frac{-\omega^2}{(1 - \omega^2/\omega_2^2)^2}}; \qquad \omega_2^2 = \frac{1}{m_1 m_2}. \tag{28}$$

Band Elimination Filter

For a two-peak band elimination filter such as shown in Fig. 3, filter 2, the equations are:

$$Z_0 = \sqrt{\frac{L_1}{C_2}\frac{(1 - \omega^2/\omega_A^2)(1 - \omega^2/\omega_B^2)}{(1 - \omega^2/\omega_2^2)^2}};$$
$$\tanh\frac{P}{2} = \frac{1}{\omega_a}\sqrt{\frac{-\omega^2}{(1 - \omega^2/\omega_A^2)(1 - \omega_2^2/\omega_B^2)}}; \tag{29}$$
$$\omega_a = \frac{1}{\sqrt{L_1 C_2}} = \sqrt{\frac{-\omega_{\infty 1}^2}{(1 - \omega_{\infty 1}^2/\omega_A^2)(1 - \omega_{\infty 1}^2/\omega_B^2)}},$$

$\omega_{\infty 1}\omega_{\infty 2} = \omega_A \omega_B.$

In a similar manner more sections can be added and the resonant frequencies determined in terms of the cutoff frequencies and the position of the attenuation peaks. The most general section considered in this paper has a maximum of five equivalent sections. For this case by applying the process described above the propagation constant and critical frequencies are given by the equations

$$\tanh\frac{P}{2} = \frac{A + C + E}{1 + B + D}\sqrt{\frac{(1 - \omega^2/\omega_A^2)(1 - \omega^2/\omega_3^2)^2(1 - \omega^2/\omega_5^2)^2}{(1 - \omega^2/\omega_2^2)^2(1 - \omega^2/\omega_4^2)^2(1 - \omega^2/\omega_B^2)}}, \quad (30)$$

where

$$A = \sum_1^5 m = m_1 + m_2 + m_3 + m_4 + m_5;$$

$$B = \sum_{m=1}^{5}\sum_{n=1}^{5} m_m m_n; \qquad n \neq m;$$

$$C = \sum_{m=1}^{5}\sum_{n=1}^{5}\sum_{o=1}^{5} m_m m_n m_o; \qquad n \neq m; \qquad n \neq o; \qquad m \neq o;$$

$$D = \sum_{m=1}^{5}\sum_{n=1}^{5}\sum_{o=1}^{5}\sum_{p=1}^{5} m_m m_n m_o m_p; \qquad m \neq n; \qquad m \neq o;$$

$$m \neq p; \qquad n \neq p; \qquad n \neq o; \qquad o \neq p;$$

$$E = m_1 m_2 m_3 m_4 m_5. \quad (31)$$

The resonant frequencies are given by the equations

$$f_2^2 = \frac{2f_A^2 f_B^2(1 + B + D)}{f_A^2(2 + B - \sqrt{B^2 - 4D}) + f_B^2(B + 2D + \sqrt{B^2 - 4D})}, \quad (32)$$

$$f_4^2 = \frac{2f_A^2 f_B^2(1 + B + D)}{f_A^2(2 + B + \sqrt{B^2 - 4D}) + f_B^2(B + 2D - \sqrt{B^2 - 4D})}, \quad (33)$$

$$f_3^2 = \frac{2f_A^2 f_B^2(A + C + E)}{f_A^2(2A + C - \sqrt{C^2 - 4AE}) + f_B^2(C + 2E + \sqrt{C^2 - 4AE})}, \quad (34)$$

$$f_5^2 = \frac{2f_A^2 f_B^2(A + C + E)}{f_A^2(2A + C + \sqrt{C^2 - 4AE}) + f_B^2(C + 2E - \sqrt{C^2 - 4AE})}. \quad (35)$$

For any smaller number of sections the values can be obtained by letting some of the m's go to zero. For example, for a three section filter $m_4 = m_5 = 0$. For low, high, and all-pass networks the values can be obtained by letting $f_A^2 \to 0$, $f_B^2 \to \infty$ or $f_A^2 \to 0; f_B^2 \to \infty$.